DETERMINATION OF THERMAL CONDUCTIVITY AND DIFFUSIVITY FROM MEASUREMENTS OF UNSTEADY TEMPERATURES

Yu. L. Gur'ev and D. N. Chubarov

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Explicit formulas are presented for determining the thermal conductivity and diffusivity from measurements of unsteady temperatures in various shaped samples.

Explicit expressions for determining the thermal diffusivity from measurements of unsteady temperatures in various shaped samples were derived in [1] by using Laplace transforms. The same approach can be used to construct explicit relations for the simultaneous determination of the thermal conductivity and diffusivity from measurements of unsteady temperatures for a known heat load (heat-flux densities) on the test sample.

Suppose we know the law of variation of temperature of an unbounded flat plate when one surface receives a known heat flux and the other surface is thermally insulated. The Laplace transform of the relation between the temperature at the point $x = \delta$ and the heat flux q(t) at $x = \delta$ has the form [2]

$$\frac{q(s)}{\sqrt{s}\,T(\delta,\ s)} = \frac{\lambda}{\sqrt{a}}\,\mathrm{th}\sqrt{\frac{s}{a}}\,\delta. \tag{1}$$

Differentiation of (1) with respect to s gives

$$\frac{q'(s)\,T(\delta,\ s) - q(s)\,T'(\delta,\ s)}{\sqrt{s}\,T^2(\delta,\ s)} - \frac{q(s)}{2s\sqrt{s}\,T(\delta,\ s)} = \frac{\lambda\delta}{2a\sqrt{s}}\left(1 - \mathrm{th}^2\sqrt{\frac{s}{a}}\,\delta\right). \tag{2}$$

Using the fact that $q(s) = \lambda\sqrt{s/a}\,T(\delta,\ s)\tanh\sqrt{s/a}\,\delta$, we obtain from (2)

$$q'(s)\,T(\delta,\ s) - q(s)\,T'(\delta,\ s) - \frac{1}{2s}\,q(s)\,T(\delta,\ s) = \frac{\lambda\delta}{2a}\,T^2(\delta,\ s) - \frac{\delta}{\lambda}\,\frac{q^2(s)}{2s}. \tag{3}$$

Taking the inverse transform of (3) by using known inversion formulas [3], we obtain

$$\frac{\lambda^2\delta}{2a}\,\psi(t) - \varphi(t)\,\lambda - \frac{\delta}{2}\,\gamma(t) = 0, \tag{4}$$

from which

$$\lambda = \frac{a}{\delta}\,\frac{\varphi(t)}{\psi(t)} + \sqrt{\left[\frac{a\varphi(t)}{\delta\psi(t)}\right]^2 + \frac{a\gamma(t)}{\psi(t)}}. \tag{5}$$

Here ψ, φ, and γ represent the following integral combinations:

$$\psi(t) = \int_0^t T(\delta,\ t-\tau)\,T(\delta,\ \tau)\,d\tau, \tag{5'}$$

$$\varphi(t) = \int_0^t \left\{[q(t-\tau)\,T(\delta,\ \tau) - q(\tau)\,T(\delta,\ t-\tau)]\,\tau - fT(\delta,\ t-\tau)\int_0^\tau q(\theta)\,d\theta\right\}d\tau, \tag{5''}$$

$$\gamma(t) = \int_0^t q(t-\tau)\int_0^\tau q(\theta)\,d\theta d\tau,\quad f = 1/2. \tag{5'''}$$

Equation (5) permits the determination of the thermal conductivity if the thermal diffusivity a is known, which is not always the case. Since the values of a and λ are assumed constant in the temperature range under investigation, Eq. (5) must give the same values at different times for the same T(δ, t) and q(t), i.e., $\lambda_i = \lambda_j$ and $a_i = a_j$, or at any times for two different T(δ, t) and q(t). Taking this into account we find from (5)

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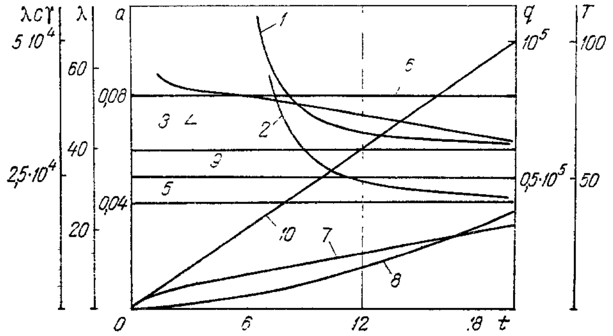

Fig. 1. Calculated values of coefficients: 1) λ, W/m·°K (calculated value); 2) a, m^2/h; 3) λcγ, W^2·h/m^4·°K (semibounded body); 4-6) assumed values of λ, a, and λcγ, respectively; 7, 8) surface temperature of sample in two numerical experiments, °K; 9, 10) thermal flux densities on the surface of sample in two numerical experiments, W/m^2; t is in sec.

$$\lambda=\frac{\delta_j}{2}\cdot\frac{\psi_j\gamma_i}{\psi_i\varphi_j}\cdot\frac{\left[1-\frac{\psi_i\gamma_j}{\psi_j\gamma_i}\right]}{\left[1-\frac{\delta_j}{\delta_i}\,\frac{\psi_j\varphi_i}{\psi_i\varphi_j}\right]}, \tag{6}$$

$$a=\frac{\lambda^2\delta_i\psi_i}{2\left[\lambda\varphi_i-\frac{\delta_i}{2}\,\gamma_i\right]} \quad \text{or} \quad a=\frac{\lambda^2\delta_j\psi_j}{2\left[\lambda\varphi_j+\frac{\delta_j}{2}\,\gamma_j\right]}. \tag{7}$$

In Eqs. (6) and (7) the subscripts i and j refer either to different times and the same T(δ, t) and q(t), or to arbitrary times and two different T(δ. t) and q(t). where in the first case $\delta_i=\delta_j$, while in the second case δ_i may or may not be equal to δ_j. Depending on the availability and quality of the experimental data, either of the above schemes can be used to calculate the parameters.

Because of a certain complication of the final expressions there is a possibility of determining λ without first calculating a or using the procedure described above. It follows from (1) that

$$\left\{\left[\frac{q(s)}{\sqrt{s}\,T(\delta,\ s)}\right]'\sqrt{s}\right\}'=-\frac{\delta^2\lambda\,\mathrm{sh}\sqrt{s/a}\,\delta}{2a\sqrt{sa}\,\mathrm{ch}^3\sqrt{s/a}\,\delta}. \tag{8}$$

Denoting $T^3(\delta,\ s)$ times the left-hand side of (8) by F(s), making some simple substitutions, and using (1) and (2), we obtain

$$-sF(s)=\frac{\delta}{\lambda}\,q(s)\left[q'(s)\,T(\delta,\ s)-q(s)\,T'(\delta,\ s)-\frac{1}{2}\,\frac{q(s)\,T(\delta,\ s)}{s}\right],$$

$$sF(s)=sq''(s)\,T^2(\delta,\ s)-sq(s)\,T''(\delta,\ s)\,T(\delta,\ s)-2sq'(s)\,T'(\delta,\ s)\,T(\delta,\ s)+2sq(s)\,T'^2(\delta,\ s)$$
$$-\frac{1}{2}\left[q'(s)\,T^2(\delta,\ s)-q(s)\,T'(\delta,\ s)\,T(\delta,\ s)-\frac{q(s)\,T^2(\delta,\ s)}{s}\right]. \tag{9}$$

After taking inverse transforms it follows from (9) that

$$\lambda=\delta\,\frac{\varphi_1(t)}{\psi_1(t)}, \tag{10}$$

where φ_1 and ψ_1 represent the following integral combinations:

$$\varphi_1(t)=\int_0^t q(t-\tau)\int_0^\tau\left\{\theta\,[T(\delta,\ \theta)\,q(\tau-\theta)-q(\theta)\,T(\delta,\ \tau-\theta)]-\frac{1}{2}\,T(\delta,\ \tau-\theta)\int_0^\theta q(\xi)\,d\xi\right\}d\theta d\tau, \tag{10'}$$

$$\psi_1(t)=\int_0^t T(\delta,\ t-\tau)\int_0^\tau\left\{(\tau-\theta)^2\left[T(\delta,\ \tau-\theta)\,\frac{dq(\theta)}{d\theta}-q(\tau-\theta)\,\frac{dT(\delta,\ \theta)}{d\theta}\right]\right.$$
$$\left.-\frac{1}{2}\,\theta\,[q(\theta)\,T(\delta,\ \tau-\theta)-T(\delta,\ \theta)\,q(\ \ -\theta)]+\frac{1}{2}\,T(\delta,\ \tau-\theta)\int_0^\theta q(\xi)\,d\xi\right\}d\theta d\tau+2\int_0^t(t-\tau)\,T(\delta,\ t-\tau)$$

$$\times \int_0^\tau (\tau-\theta)\left[q(\tau-\theta)\frac{dT(\delta,\theta)}{d\theta}-T(\delta,\tau-\theta)\frac{dq(\theta)}{d\theta}\right]d\theta d\tau. \tag{10''}$$

Thus, from a measurement of the surface temperature of a flat plate and the heat flux incident upon it, Eq. (10) can be used to determine the thermal conductivity without considering and analyzing the solution of the forward heating problem. Of course, it is easy to find the diffusivity from (7) after determining the thermal conductivity from (10).

A similar procedure can be used to obtain explicit relations when the temperature is measured on the thermally insulated surface of the sample. In this case the starting point is the relation between the Laplace transforms of q(t) and T(0, t)

$$q(s)=\lambda\sqrt{s/a}\,\mathrm{sh}\sqrt{s/a}\,\delta T(0,s). \tag{11}$$

It follows from (11) that

$$\left[\frac{\sqrt{s}T(0,s)}{q(s)}\right]'\frac{\lambda}{\sqrt{a}}=-\frac{\delta\,\mathrm{ch}\sqrt{s/a}\,\delta}{2\sqrt{sa}\,\mathrm{sh}^2\sqrt{s/a}\,\delta}. \tag{12}$$

After multiplying both sides of this equation by $2\sqrt{sa/\delta}$ and differentiating with respect to s, we obtain

$$\frac{4\lambda}{\delta^2}\sqrt{sa}\,F_1(s)=\frac{1}{\mathrm{sh}\sqrt{s/a}\,\delta}+\frac{2}{\mathrm{sh}^3\sqrt{s/a}\,\delta}, \tag{13}$$

where

$$F_1(s)=\left\{\left[\frac{\sqrt{s}T(0,s)}{q(s)}\right]'\sqrt{s}\right\}'.$$

After multiplying both sides of (13) by $q^3(s)/(s/a)\sqrt{s/a}$ and using (11) we find

$$\frac{2a^2}{\delta^2}F_2(s)=\lambda^2 sT^3(0,s)+\frac{a}{2}q^2(s)T(0,s), \tag{14}$$

$$F_2(s)=\frac{T'(0,s)q^2(s)-T(0,s)q'(s)q(s)}{2}$$
$$-q(s)[T(0,s)q'(s)-T'(0,s)q(s)]$$
$$+sq(s)[T''(0,s)q(s)-T(0,s)q''(s)]$$
$$+2sq'(s)[T(0,s)q'(s)-T'(0,s)q(s)].$$

Taking the inverse transforms of (14) we obtain

$$\lambda=\sqrt{\frac{2a^2}{\delta^2}\frac{\varphi_2(t)}{\psi_2(t)}-\frac{a}{2}\frac{\gamma_2(t)}{\psi_2(t)}}, \tag{15}$$

where φ_2, ψ_2, and γ_2 represent the following integral combinations:

$$\psi_2(t)=\int_0^t T(0,t-\tau)\int_0^\tau T(0,\tau-\theta)\frac{dT(0,\theta)}{d\theta}d\theta d\tau, \tag{15'}$$

$$\gamma_2(t)=\int_0^t T(0,t-\tau)\int_0^\tau q(\tau-\theta)q(\theta)\,d\theta d\tau, \tag{15''}$$

$$\varphi_2(t)=\int_0^t q(t-\tau)\int_0^\tau\Big\{\frac{3}{2}\theta[q(\theta)T(0,\tau-\theta)-T(0,\theta)q(\tau-\theta)]$$
$$+(\tau-\theta)^2\left[T(0,\tau-\theta)\frac{dq(\theta)}{d\theta}-q(\tau-\theta)\frac{dT(0,\theta)}{d\theta}\right]\Big\}d\theta d\tau+2\int_0^t(t-\tau)q(t-\tau)\int_0^\tau(\tau-\theta)$$
$$\times\left[q(\tau-\theta)\frac{dT(0,\theta)}{d\theta}-T(0,\tau-\theta)\frac{dq(\theta)}{d\theta}\right]d\theta d\tau. \tag{15'''}$$

If a is not known beforehand, the value of the thermal diffusivity in Eq. (15) can be determined by the method described above, equating the values of λ in Eq. (15) for two different times and the same T(δ, t) and q(t), or for two arbitrary times and different T(δ, t) and q(t). Then

$$a = \frac{\delta_i^2}{4}\left(\frac{\gamma_i}{\psi_i} - \frac{\gamma_j}{\psi_j}\right)\left(\frac{\varphi_i}{\psi_i} - \frac{\varphi_j}{\psi_j}\frac{\delta_i^2}{\delta_j^2}\right)^{-1}, \tag{16}$$

where the subscripts i and j correspond to values of the integral combinations calculated for two different times and the same $T(\delta, t)$ and $q(t)$ or for arbitrary times and two different $T(\delta, t)$ and $q(t)$. The thermal diffusivity can be determined another way also. It follows from (11) that

$$\left\{\left[\frac{q(s)}{\sqrt{s}T(0, s)}\right]' \sqrt{s}\right\}' = \frac{\lambda\delta^2}{4a\sqrt{sa}} \operatorname{sh} \sqrt{\frac{s}{a}}\,\delta.$$

After differentiating the left-hand side of this expression and using (11), we obtain

$$a = \frac{\delta^2}{4}\,\frac{\varphi_3(t)}{\psi_3(t)}. \tag{17}$$

Here φ_3 and ψ_3 are determined from the following relations:

$$\varphi_3(t) = \int_0^t T(0, t-\tau)\int_0^\tau q(\theta)\, T(0, \tau-\theta)\, d\theta d\tau, \tag{17'}$$

$$\psi_3(t) = \int_0^t T(0, t-\tau)\int_0^\tau \Big\{(\tau-\theta)^2\left[q(\tau-\theta)\frac{dT(0, \theta)}{d\theta} - T(0, \tau-\theta)\frac{dq(\theta)}{d\theta}\right]$$
$$+\frac{1}{2}\theta\,[T(0, \tau-\theta)\, q(\theta) - q(\tau-\theta)\, T(0, \theta)]$$
$$+\frac{1}{2}T(0, \tau-\theta)\int_0^\theta q(\xi)\, d\xi\Big\} d\theta d\tau + 2\int_0^t (t-\tau)\, T(0, t-\tau)$$
$$\times\int_0^\tau \Big\{(\tau-\theta)\left[T(0, \tau-\theta)\frac{dq(\theta)}{d\theta} - q(\tau-\theta)\frac{dT(0, \theta)}{d\theta}\right]\Big\} d\theta d\tau. \tag{17''}$$

Similar calculations lead to relations for determining λ and a when the temperature at the point $x = \delta/2$ is known. Omitting the intermediate calculations, we obtain finally

$$\lambda = \frac{1}{2}\sqrt{\frac{2a^2\varphi(t)}{(\delta/2)^2\,\psi(t)} - \frac{a\gamma(t)}{2\psi(t)}}, \tag{18}$$

where φ, ψ, and γ are determined from Eqs. (15')-(15''') with $T(\delta/2, t)$ substituted for $T(0, \tau)$. The value of the thermal diffusivity which is needed to calculate λ from (18) can be determined either from Eq. (16) with $\delta_i/2$ substituted for δ_i, or from the expression

$$a = \frac{(\delta/2)^2}{4}\cdot\frac{\varphi(t)}{\psi(t)}. \tag{19}$$

The values of φ and ψ represent the same integral combinations as (17') and (17'') with $T(\delta/2, t)$ substituted for $T(0, t)$. Equation (19) is obtained in the same way as (17) by using the relation between the Laplace transforms of $q(t)$ and $T(\delta/2, t)$.

Calculations similar to those presented above performed for a solid cylinder and sphere with a known surface temperature and heat flux on the surface lead to relations of the form (5) and (6). In these equations δ must be replaced by the radius of the cylinder or sphere, $f = {}^1/_2$ for a cylinder and $f = -{}^1/_2$ for a sphere.

Thus, from the relations obtained, the values of the thermal conductivity and diffusivity can be determined simultaneously from a measurement of unsteady temperatures in various shaped samples. In contrast with existing methods, the required parameters are expressed explicitly in terms of experimentally known temperatures and heat fluxes. The integral combinations entering the computing formulas are easily calculated on a computer without constructing complicated algorithms. Figure 1 shows the results of computer calculations of the thermophysical parameters of a flat plate thermally insulated from one side. The "experimental" values of the temperatures were obtained by solving the forward problem for heat-flux densities which are constant and vary linearly with time. The plate was 10 mm thick, had a thermal conductivity $\lambda = 40$ W/m · °K, and a thermal diffusivity $a = 0.04$ m^2/h. The data in Fig. 1 show that the calculated values of λ and a converge rather rapidly to the values assumed to solve the forward heating problem. It should be noted that the large difference between the calculated and assumed values of λ and a at early times result from the fact that at small values of t the sample behaves thermally like a semibounded body whose surface temperature is determined by the combination $\lambda c\gamma$ and not by λ and a separately. Therefore, with the calculational accuracy which can be achieved in practice it is impossible in the initial period of heating or cooling to reveal the finite thickness of the sample which is necessary for successful calculation with Eqs. (6) and (7). On the other hand, the determination of the parameter $\lambda c\gamma$ from the expression

$$\lambda c\gamma = \frac{\gamma_i}{\psi_i}, \tag{20}$$

which is easily obtained from the relation between Laplace transforms for a semibounded body $[q(s)/T(s)]^2 = s(\lambda c\gamma)$, shows that the calculated values of this parameter in the initial period of heating or cooling are in good agreement with the values assumed in the solution of the forward problem. Thus, the simultaneous calculation of $\lambda c\gamma$ by Eq. (20) and λ and a by (6) and (7) in the present case enables one to judge from the nature of the variation of these parameters both the accuracy of the computational scheme chosen and the legitimacy of using the recommended relations as a whole for the available experimental data.

NOTATION

T, temperature of sample; x, coordinate; δ, thickness of flat plate; λ, thermal conductivity; a, thermal diffusivity; c, specific heat; γ, density; q, heat flux density; t, τ, θ, time.

LITERATURE CITED

1. D. N. Chubarov, Inzh.-Fiz. Zh., 32, No. 3 (1977).
2. A. V. Lykov, The Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
3. V. A. Ditkin and A. P. Prudnikov, Integral Transforms and Operational Calculus, Oxford Univ. Press, New York (1965).

PHOTOEMISSION METHOD OF MEASURING TEMPERATURE

K. N. Kasparov

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A photoemission method of measuring temperature is presented, and the range of its application is indicated. Expressions are obtained for calculating the systematic error, and a nomogram is given for determining it.

The photoemission method of measuring temperature for a continuous emission spectrum is based on the dependence of the energy distribution of photoelectrons in the photoemission effect on the energy distribution in the spectrum of the radiation source [1, 2]. The temperature of a body is determined from the change in the energy distribution of photoelectrons, i.e., the increase in the number of photoelectrons with the maximum kinetic energy $W_{max} = eU_{max}$ with increasing temperature.

By considering Einstein's equation for the photoelectric effect

$$eU_{max} = h(\nu - \nu_0) \tag{1}$$

or

$$W_{max} = f_1(\nu)$$

together with Planck's equation for blackbody radiation

$$r_0 = f_2(\nu, T) \tag{2}$$

it is clear that an implicit relation exists between the maximum kinetic energy of the photoelectrons and the temperature T of the body whose radiation gave rise to the photoelectric effect. It follows from (2) that the frequency ν is a function of the spectral density of the blackbody radiation energy r_0 and the temperature

$$\nu = \varphi(r_0, T) \tag{3}$$

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